

Massive Quark-Gluon Scattering Amplitudes at Tree Level

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Abstract

Results for four-, five-, and six-parton tree amplitudes for massive quark-antiquark scattering with gluons are calculated using the recursion relations of Britto, Cachazo, Feng, and Witten. The required diagrams are generated using shifts of the momenta of a pair of massless legs to complex values. Checks verifying the calculations are described, and a simple formula for the shifted spinors of an internal gluon is presented.

I. INTRODUCTION

Stimulated by Witten's introduction of a string theory on twistor space dual to perturbative gauge theory [1], significant recent progress has been made in the calculation and understanding of the structure of gauge-theory amplitudes. By revealing the maximally helicity violating amplitudes to be primitive vertices from which an amplitude can be computed, Feynman diagram calculations can be replaced by the MHV rules of Cachazo, Svrček, and Witten [2] relating the desired tree amplitude with products of propagators and on-shell MHV amplitudes that possess fewer legs. Britto, Cachazo, Feng, and Witten (BCFW) [3, 4] later proved that a tree amplitude is given by the sum of products of a propagator and two on-shell amplitudes with fewer legs and shifted, complex momenta. This recursion yields very compact expressions for an amplitude. Recursive methods for the calculation of leading-order QCD processes have existed for many years [5, 6], initially formulated by Berends and Giele. In contrast to the Berends-Giele recursion relation which is based on off-shell vertices, the building blocks for the new methods are on-shell tree amplitudes. On-shell methods have been previously used at loop level via the unitarity method [7].

The BCFW recursion, while initially derived for pure gauge boson tree amplitudes, has been generalized to include massive particles with spin through the work of Badger, Glover, Khoze, and Svrček [8]. Thus the new recursive methods can be brought to bear on a variety of phenomenologically interesting problems to be encountered at the Large Hadron Collider such as top quark, vector boson, and possible supersymmetric processes. The multigluon scattering with a single external massive gauge boson or Higgs boson has been calculated [9, 10, 11, 12] as well as processes with a pair of massive scalar particles [8, 13]. The supersymmetry Ward identities of supersymmetric QCD have been applied to these scalar interactions, yielding the quark amplitudes for certain helicities [14]. Besides tree-level gauge theory [8, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], the BCFW recursion has been utilized for one-loop QCD calculations [27, 28, 29, 30, 31, 32, 33, 34] and gravity [35, 36, 37, 38, 39].

The present paper is concerned with the tree-level scattering of a massive fermion pair with massless gauge bosons. Results for $\bar{q}q \rightarrow ggg$ were presented by Ozeren and Stirling, but these authors report that difficulties are encountered with the BCFW recursion so that Feynman diagrams were necessary when gluons are exchanged for certain helicity and spin

configurations [21]. Here we remedy this and present an explicit formula for the spinors associated with the massless gluon exchanged within a diagram. In this paper, the massive quark-gluon tree amplitudes with six and fewer partons are presented. The necessary recursion diagrams are found solely via the BCFW method with spinor shifts on two of the external massless gluon legs. Recent work by Schwinn and Weinzierl [40] also deals with scattering of massive quarks which also avoid the potential difficulties. In that reference a formula is given for the scattering of a pair of massive quarks with an arbitrary number of gluons in a specific helicity configuration, one negative-helicity gluon and the rest positive. Here we provide all the remaining helicity configurations up to amplitudes with four external gluons.

II. REVIEW OF THE SPINOR-HELICITY FORMALISM

The tree amplitudes presented here are the color-ordered partial amplitudes A_n , containing the kinematic data, from which the full amplitude M_n with color information is determined by the color decomposition [42],

$$M_n(k_i, \lambda_i, a_i) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \dots T^{a_{\sigma(n)}})_{i_2}^{\bar{j}_1} A_n(\bar{1}_q^{\lambda_1}, 2_q^{\lambda_2}, \sigma(3^{\lambda_3})_g, \dots \sigma(n^{\lambda_n})_g). \quad (2.1)$$

The notation k^{λ_k} labels the k -th particle with spin λ_k , S_{n-2} is the group of permutations on $n-2$ labels, and the quark colors are i_2 and \bar{j}_1 for the quark labelled 2 and antiquark 1. The color generators are normalized as $\text{tr} T^a T^b = \delta^{ab}$.

In the spinor helicity formalism, spinors for the particle with momentum p_i are denoted

$$\begin{aligned} u^+(p_i) &\equiv |i\rangle, & \overline{u}^+(p_i) &\equiv [i], \\ u^-(p_i) &\equiv |i], & \overline{u}^-(p_i) &\equiv \langle i|. \end{aligned} \quad (2.2)$$

For massive spinors, the same angle and square brackets denote spin with respect to a fixed axis rather than helicity. These massive spinors have four components and satisfy the Dirac equation, and they are interpreted as Weyl spinors only in the massless limit.

Spinor products take the form

$$\begin{aligned} \overline{u}^-(p_i) u^+(p_j) &= \langle ij \rangle, & \overline{u}^+(p_i) u^-(p_j) &= [ij], \\ \overline{u}^+(p_i) u^+(p_j) &= [ij], & \overline{u}^-(p_i) u^-(p_j) &= \langle ij \rangle. \end{aligned} \quad (2.3)$$

We use the formalism of Kleiss and Stirling [41] for the construction of massive spinors, which yield

$$\begin{aligned}
\langle ij \rangle &= \frac{(p_j \cdot k_0)(p_i \cdot k_1) - (p_i \cdot k_0)(p_j \cdot k_1) - i\epsilon_{\mu\nu\rho\sigma} k_0^\mu p_i^\nu p_j^\rho k_1^\sigma}{\sqrt{(p_i \cdot k_0)(p_j \cdot k_0)}}, \\
[ij] &= \frac{(p_i \cdot k_0)(p_j \cdot k_1) - (p_j \cdot k_0)(p_i \cdot k_1) - i\epsilon_{\mu\nu\rho\sigma} k_0^\mu p_i^\nu p_j^\rho k_1^\sigma}{\sqrt{(p_i \cdot k_0)(p_j \cdot k_0)}}, \\
[ij] &= \frac{m_i(p_j \cdot k_0) + m_j(p_i \cdot k_0)}{\sqrt{(p_i \cdot k_0)(p_j \cdot k_0)}}, \tag{2.4}
\end{aligned}$$

where k_0 and k_1 are vectors satisfying $k_0^2 = 0$, $k_1^2 = -1$, and $k_0 \cdot k_1 = 0$. The vector k_0 corresponds to the axis of spin quantization for the massive fermions. Implicit in these spinor-product formulas is their analytic continuation under crossing symmetry. A past-pointing momentum p_i is replaced with $-p_i$ while each formula acquires an overall factor of i , and m_i becomes $-m_i$ ($m > 0$ for all legs).

The final spinor product required is $\langle i|\not{p}|j\rangle$, whose formula is determined by the spin-sum rule for a particle with momentum p , $|p\rangle[p] + |p\rangle\langle p| = \not{p} + m_p$, so that

$$\langle i|\not{p}|j\rangle = \langle ip\rangle[pj] + \langle ip\rangle\langle pj\rangle - m_p\langle ij\rangle. \tag{2.5}$$

If p corresponds to an antiparticle, $m_p \mapsto -m_p$. The shorthand $\langle i|j|k\rangle$ is used for this spinor product. Under complex conjugation, the spinor products transform as

$$\begin{aligned}
\langle ab \rangle^* &= [ba], \\
[ab]^* &= \langle ab \rangle, \\
\langle a|k|b \rangle^* &= [a|k|b].
\end{aligned} \tag{2.6}$$

The Schouten identity for two-component spinors is

$$|a\rangle\langle bc\rangle + |c\rangle\langle ab\rangle + |b\rangle\langle ca\rangle = 0, \tag{2.7}$$

and we have spinor identities for massive legs i and j ,

$$[ab]\left(\langle ia\rangle\langle bj\rangle - \langle ib\rangle\langle aj\rangle\right) + \langle ab\rangle\left(\langle ia\rangle[bj] - \langle ib\rangle[aj]\right) = 2p_a \cdot p_b\langle ij\rangle, \tag{2.8}$$

$$m_i\langle bc\rangle + [ic]\langle ib\rangle + [ib]\langle ci\rangle = 0, \tag{2.9}$$

and

$$[ia]\langle jb\rangle = [ib]\langle ja\rangle. \tag{2.10}$$

Complex conjugation of these identities yields their square-bracket counterparts.

For helicity-labelling conventions we consider all the gluons to be outgoing so that a helicity label on a gluon denotes its helicity leaving a vertex. The massive quarks are both considered to be incoming. Thus momentum conservation means $p_1 + p_2 = p_3 + \dots + p_n$. Our fermion spin labels are therefore opposite to the helicity labels for outgoing particles in the massless limit.

Simple relations exist between n -point amplitudes with different helicities and spins. Complex conjugation of an amplitude flips the spin of every particle,

$$A_n(\bar{1}_q^{s_1}, 2_q^{s_2}, 3_g^{s_3}, \dots, n_g^{s_n})^* = A_n(\bar{1}_q^{-s_1}, 2_q^{-s_2}, 3_g^{-s_3}, \dots, n_g^{-s_n}), \quad (2.11)$$

and the spin of an individual quark may be flipped by replacing bracket spinors with angle spinors, or vice versa, so we have

$$\begin{aligned} A_n(\bar{1}_q^\pm, 2_q^{s_2}, 3_g^{s_3}, \dots, n_g^{s_n}) \Big|_{\substack{[1] \mapsto \langle 1| \\ \langle 1| \mapsto [1]}} &= A_n(\bar{1}_q^\mp, 2_q^{s_2}, 3_g^{s_3}, \dots, n_g^{s_n}), \\ A_n(\bar{1}_q^{s_1}, 2_q^\pm, 3_g^{s_3}, \dots, n_g^{s_n}) \Big|_{\substack{[2] \mapsto |2\rangle \\ |2\rangle \mapsto [2]}} &= A_n(\bar{1}_q^{s_1}, 2_q^\mp, 3_g^{s_3}, \dots, n_g^{s_n}). \end{aligned} \quad (2.12)$$

Thus, for example, complex conjugation of the amplitude $A_n(\bar{1}_q^-, 2_q^-, 3_g^{s_3}, \dots, n_g^{s_n})$ followed by the replacements $\langle 1| \mapsto [1|$ and $|2\rangle \mapsto [2]$ flips the helicity of only the gluons, yielding the amplitude $A_n(\bar{1}_q^+, 2_q^+, 3_g^{-s_3}, \dots, n_g^{-s_n})$. Explicitly following the terms $[1a]$ and $[1a\rangle$ through this composition of maps we have

$$\begin{aligned} [1a] &\xrightarrow{*} -\langle 1a \rangle \xrightarrow{\langle 1| \mapsto [1|} -[1a], \\ [1a\rangle &\xrightarrow{*} \langle 1a \rangle \xrightarrow{\langle 1| \mapsto [1|} [1a]. \end{aligned} \quad (2.13)$$

Every possible combination of spins can be obtained from the amplitudes $A_n(\bar{1}_q^-, 2_q^-, 3_g^{s_3}, \dots, n_g^{s_n})$ by a combination of complex conjugation and quark spinor replacements. Thus in presenting the results for the amplitudes we need only present the $\bar{1}_q^-, 2_q^-$ amplitudes and half of the possible gluon helicity configurations. The spinor identities (2.7), (2.8), (2.9), and (2.10) can only be used when the massive fermions have particular spins, so their use will invalidate the above relations between amplitudes. We present the amplitudes in a form where no spin-dependent identities are applied in order to allow relations (2.11) and (2.12) to be used.

The amplitudes presented in ref. [40] use a different spinor formalism which must be accounted for in comparison of the amplitudes. The spin quantization axis k_0 is chosen to be one of the massless gluon legs, labelled as leg q , and massive vectors are projected to the light cone with this leg q ,

$$k^b = k - \frac{k^2}{2k \cdot q} q. \quad (2.14)$$

Massive spinors are defined as

$$\begin{aligned} u^+(k) &= |k^b\rangle + \frac{m}{[k^b q]} |q], \\ u^-(k) &= |k^b] + \frac{m}{\langle q k^b \rangle} |q\rangle, \end{aligned} \quad (2.15)$$

which leads to the spinor products

$$\begin{aligned} \langle ij \rangle &= \langle i^b j^b \rangle, \quad [ij] = [i^b j^b], \\ [ij] &= m_i \frac{\langle qj \rangle}{\langle qi \rangle} + m_j \frac{[qi]}{[qj]}. \end{aligned} \quad (2.16)$$

III. THE BCFW RECURSION FORMULA

The BCFW tree-level recursion formula follows from the basic complex analytic properties of amplitudes in gauge theory. Consider a tree-level color-ordered gauge-theory amplitude $A_n(p_1, \dots, p_n)$ with complex momenta obtained by shifting a pair of spinors from the massless external legs i and j ,

$$\widehat{|i\rangle} \equiv |i\rangle + z|j\rangle, \quad \widehat{|j]} \equiv |j] - z|i]. \quad (3.1)$$

Under this spinor shift, the momenta shift as

$$\begin{aligned} \widehat{p}_i &= |i\rangle[i] + z|j\rangle[i], \\ \widehat{p}_j &= |j\rangle[j] - z|j\rangle[i]. \end{aligned} \quad (3.2)$$

To be concise we denote this choice of shifted legs “ $\langle ij \rangle$ ”, but from context this should not be confused with the spinor product of eq. (2.3).

This BCFW shift keeps all momenta on shell, and total momentum is still conserved,

$$\widehat{p}_i + \widehat{p}_j = p_i + p_j.$$

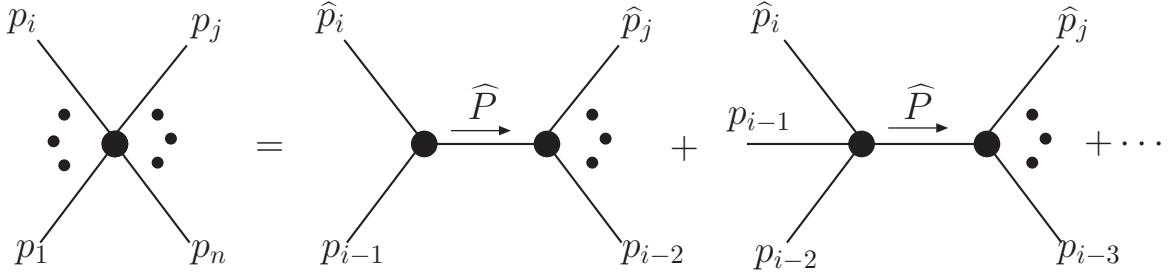


FIG. 1: An illustration of the recursion formula for cases where the shifted external legs p_i and p_j are chosen to be adjacent with $i < j$. The sum over the internal particle's helicity is implicit. We denote the BCFW diagrams, from left to right, as $\mathcal{D}_1, \mathcal{D}_2, \dots$

The amplitude is now a meromorphic function $A_n(z)$ with simple poles in z when the sum of adjacent particles' momenta, denoted \hat{P} , is z -dependent and on shell,

$$0 = \hat{P}^2 - m^2 = P^2 - m^2 + z\langle j|P|i\rangle. \quad (3.3)$$

Provided that $A_n(z)$ vanishes as $z \rightarrow \infty$, we have $\int A(z)/z = 0$. Applying Cauchy's theorem to $\int A(z)/z$ yields residues for $z = 0$ (the desired unshifted amplitude) and the values of z for which $\hat{P}^2 - m^2 = 0$. The factorization of amplitudes at these poles yields the BCFW formula [4]

$$A_n(p_1, \dots, p_n) = \sum_{\text{partitions}} \sum_h \hat{A}_{\text{left}}(p_L, \dots, \hat{P}^h) \times \frac{1}{P^2 - m_P^2} \times \hat{A}_{\text{right}}(-\hat{P}^{-h}, \dots, p_R), \quad (3.4)$$

as illustrated in fig. 1. Each term appearing in the recursion formula is one of the residues of $A(z)/z$. The momentum \hat{P} is the shifted total momentum leaving from the right amplitude's external legs, and the recursion formula sums over the helicity of the internal state with momentum P . The sub-amplitudes $\hat{A}_{\text{left/right}}$ are to be evaluated with the shifted spinors at the value of z for which $\hat{P}^2 - m^2 = 0$.

Valid recursion relations are obtained only for certain helicities of the shifted external legs, where $A_n(z)$ vanishes as $z \rightarrow \infty$. Ref. [40] derives the valid choices of legs to shift for both massive and massless particles. We shift only massless gluons in this paper, so it suffices to note that the recursion is valid for the “ $\langle ij \rangle$ ” shift in eq. (3.1) provided that the shifted legs i and j have helicities [4]

$$(h_i, h_j) = (+, +), (+, -), \text{ or } (-, -). \quad (3.5)$$

By applying the spin-sum rule to combine the spin states of an internal fermion, rather than considering spin states separately, a term with a fermionic-particle pole in the recursion formula can instead be written as

$$\widehat{A}_{\text{left}}(p_L, \dots, \widehat{P}^*) \times \frac{\widehat{P} + m}{P^2 - m_P^2} \times \widehat{A}_{\text{right}}(-\widehat{P}^*, \dots, p_R), \quad (3.6)$$

where P^* denotes that the internal spinors have been stripped off the amplitudes. We use this method, from ref. [18], to calculate the BCFW diagrams where an internal fermion is exchanged between trees. An amplitude which is to be fed into the recursion should not be simplified in advance using any identities which depend on the spin of the massive legs. Doing so violates the rule that both spin states of an internal fermion should be summed over, not just the spin for which a particular identity holds.

Consider the spinors $|\widehat{P}\rangle$ and $|\widehat{P}]$ which appear in the recursion diagrams for the exchange of a massless particle. Because the internal particle appears with opposite helicity at each sub-amplitude, the overall amplitude will have degree zero in the \widehat{P} spinors. Thus identities such as $\langle a\widehat{P}|\widehat{P}b\rangle = \langle a|\widehat{P}|b\rangle$ may be used to eliminate the individual spinors. However, to facilitate computer programmable calculations of the diagrams, it is convenient to have a direct substitution for $|\widehat{P}\rangle$ and $|\widehat{P}]$. Moreover, an explicit formula for the \widehat{P} spinors solves the problem of dealing with massive spinor products such as $[1\widehat{P}]$ encountered in ref. [21]. To achieve this, we use the two-dimensionality of massless spinors to write the \widehat{P} spinors in terms of the shifted legs i and j (the shift is “ $\langle ij \rangle$ ”),

$$|\widehat{P}\rangle = \alpha|i\rangle + \beta|j\rangle, \quad |\widehat{P}] = \gamma|i] + \delta|j]. \quad (3.7)$$

Then by using the identities

$$\begin{aligned} \langle \widehat{P}|\widehat{P}|i\rangle &= \langle j|\widehat{P}|\widehat{P}] = 0, \\ \langle j|\widehat{P}|i\rangle &= \langle j|P|i], \end{aligned} \quad (3.8)$$

we find

$$\begin{aligned} \widehat{P} &= |\widehat{P}\rangle[\widehat{P}] = \frac{\langle j|P|i]}{2p_i \cdot p_j} \left(|i\rangle - \frac{2p_i \cdot P}{\langle j|P|i]} |j\rangle \right) \left(\frac{2p_j \cdot P}{\langle j|P|i]} [i] - [j] \right) \\ &= \frac{1}{\langle j|P|i]} (P|i]) (\langle j|P), \end{aligned} \quad (3.9)$$

providing a simple formula for the shifted spinors of an internal gluon. The overall prefactor may be associated with either $|\widehat{P}\rangle$ or $|\widehat{P}]$. This expression for \widehat{P} is valid no matter on which side, left or right, the legs i and j lie.

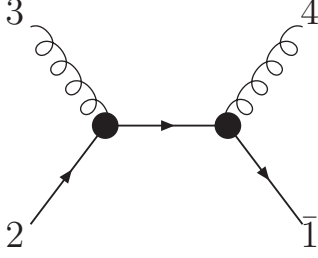


FIG. 2: The single BCFW diagram contributing to all the four-point amplitudes.

IV. $\bar{q}qgg$ AMPLITUDES

As input for the single BCFW diagram of fig. 2 contributing in the amplitudes with two gluons we use the polarization vectors of positive- and negative-helicity gluons with momentum p ,

$$\begin{aligned}\epsilon^+(p, q) &= \frac{\sqrt{2}}{\langle qp \rangle} \left(|q\rangle [p] + |p\rangle \langle q| \right), \\ \epsilon^-(p, q) &= \frac{\sqrt{2}}{[qp]} \left(|p\rangle [q] + |q\rangle \langle p| \right),\end{aligned}\tag{4.1}$$

with the three-vertex factor $\frac{1}{\sqrt{2}}$. The spinors q refer to an arbitrary null vector. Different choices of reference spinors lead to gauge-equivalent polarization vectors.

The four-point amplitudes with opposite-helicity gluons are

$$A_4(\bar{1}_q^-, 2_q^-, 3_g^+, 4_g^-) = \langle 4|1|3\rangle \frac{[13]\langle 42\rangle + [14]\langle 32\rangle}{4p_3 \cdot p_4 p_1 \cdot p_4}\tag{4.2}$$

and

$$A_4(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+) = [4|1|3\rangle \frac{[13][42] + [14]\langle 32\rangle}{4p_3 \cdot p_4 p_1 \cdot p_4}.\tag{4.3}$$

The four-point amplitudes with identical-helicity gluons are given by

$$A_4(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-) = m\langle 43\rangle \frac{[13][42] - [14][32]}{[34]^2 2p_2 \cdot p_3}\tag{4.4}$$

and

$$A_4(\bar{1}_q^-, 2_q^-, 3_g^+, 4_g^+) = m[34] \frac{[13]\langle 42\rangle - [14]\langle 32\rangle}{\langle 34\rangle^2 2p_2 \cdot p_3}.\tag{4.5}$$

Using eq. (2.10) this amplitude vanishes but only for this specific choice of quark spins. To obtain the amplitudes with the other spins, one starts from eq. (4.5) and applies eqs. (2.11)

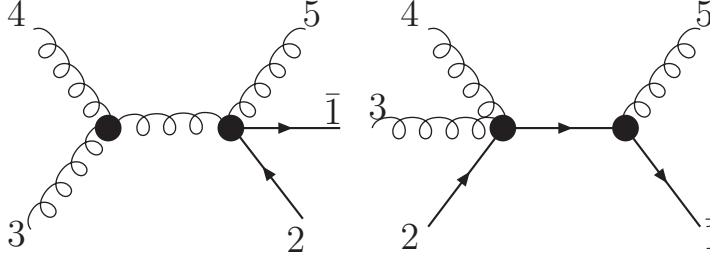


FIG. 3: The BCFW recursion diagrams required for the five-point amplitudes under the shifts “ $\langle 45 \rangle$ ” or “ $\langle 54 \rangle$ ”. These diagrams are relevant for the gluon helicities $(---)$ and $(+--)$.

and (2.12). One must include this amplitude in this form as input to higher-point recursion diagrams with internal quarks, since the internal quark is summed over all spins and not only the spin for which this amplitude happens to vanish.

Likewise note that the Schouten identity could be applied in eq. (4.4) to yield $[13][42] - [14][32] = [12][43]$, thus cancelling the unphysical double pole $[34]^2$ in the denominator. However this identity can be applied only for the particular quark spins chosen, so we leave the expression as is to avoid invalidating the next level of recursion.

V. $\bar{q}qggg$ AMPLITUDES

We now write the five-point amplitudes beginning with $A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^-)$ calculated from the shift choice “ $\langle 45 \rangle$ ”, with the result

$$A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^-) = m \left(\langle 34 \rangle + \frac{2p_1 \cdot p_5}{[4|1|5]} \langle 35 \rangle \right) \times \left[\frac{m[15][43][42] - [14][3|1|5][42] + [14][4|1|5][32]}{2p_2 \cdot p_3 [34]^2 [45] 2p_1 \cdot p_5} \right]. \quad (5.1)$$

With the shift “ $\langle 45 \rangle$ ” we find

$$A_5(\bar{1}_q^-, 2_q^-, 3_g^+, 4_g^-, 5_g^-) = \left([3|2|4] + \frac{2p_1 \cdot p_5}{[4|1|5]} [3|2|5] \right) \times \left[\frac{(-[14][3|1|5] + m[15][43])(\langle 42 \rangle + \frac{2p_1 \cdot p_5}{[4|1|5]} \langle 52 \rangle) + (-[15]2p_2 \cdot p_3 + m[14]\langle 54 \rangle)[32]}{8p_2 \cdot p_3 (p_3 \cdot p_4 + \frac{p_1 \cdot p_5}{[4|1|5]} [4|3|5]) p_1 \cdot p_5 [45]} \right] + \frac{m\langle 45 \rangle^3}{\langle 43 \rangle (p_1 + p_2)^4 (2p_1 \cdot p_5 + \frac{\langle 34 \rangle}{\langle 35 \rangle} \langle 5|1|4])} \times \left[[1\hat{P}_R][\hat{5}2] - [1\hat{5}][\hat{P}_R 2] \right]. \quad (5.2)$$

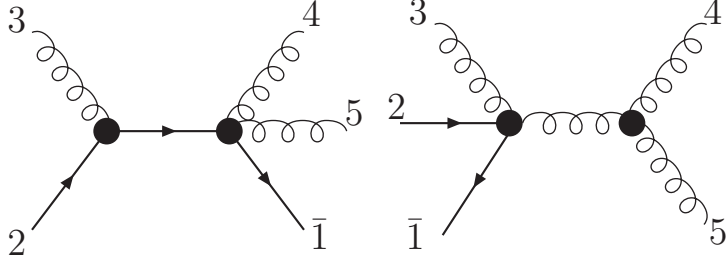


FIG. 4: The BCFW recursion diagrams required for the five-point amplitudes under the shifts “ $\langle 34 \rangle$ ” or “ $\langle 43 \rangle$ ”. These diagrams are relevant for the gluon helicities $(-+-)$ and $(-- +)$.

For the second term, which arises from the first BCFW diagram in fig. 3, we have $z = -\frac{\langle 34 \rangle}{\langle 35 \rangle}$ so that

$$\begin{aligned}\widehat{|4\rangle} &= |4\rangle - \frac{\langle 34 \rangle}{\langle 35 \rangle} |5\rangle = \frac{\langle 45 \rangle}{\langle 35 \rangle} |3\rangle, \\ \widehat{|5\rangle} &= |5\rangle + \frac{\langle 34 \rangle}{\langle 35 \rangle} |4\rangle, \\ \widehat{P}_R &= p_3 + p_4 = |3\rangle[3] + \widehat{|4\rangle}[4] = |3\rangle \left([3] + \frac{\langle 45 \rangle}{\langle 35 \rangle} [4] \right),\end{aligned}\tag{5.3}$$

where we have used the Schouten identity. Thus, for example, $[a\widehat{P}_R] = [a3] + \frac{\langle 45 \rangle}{\langle 35 \rangle} [a4]$. We use this notation throughout the paper, indicating the spinors to be substituted for the internal massless momenta \widehat{P} or \widehat{P}_R by explicitly writing $\widehat{\not{P}}$ as a bi-spinor.

The shift “ $\langle 43 \rangle$ ” leads to

$$\begin{aligned}A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+, 5_g^-) &= [4|1|5\rangle \left[\frac{m[14]\langle 53 \rangle[42] + [15]\langle 42 \rangle[4|2|3\rangle - [14]\langle 32 \rangle[4|1|5\rangle}{8p_1 \cdot p_5 p_2 \cdot p_3 \left(p_4 \cdot p_5 + \frac{p_2 \cdot p_3}{[4|2|3\rangle} [4|5|3\rangle \right) [34]} \right] \\ &+ \frac{m\langle 35 \rangle^4}{\langle 45 \rangle \langle 34 \rangle (p_1 + p_2)^4 (2p_2 \cdot p_3 + \frac{\langle 45 \rangle}{\langle 35 \rangle} \langle 3|2|4\rangle)} \times \left[[1\widehat{3}][\widehat{P}2] - [1\widehat{P}][\widehat{3}2] \right],\end{aligned}\tag{5.4}$$

where, in the second term, which arises from the first BCFW diagram in fig. 4, $z = -\frac{\langle 45 \rangle}{\langle 35 \rangle}$, which means

$$\begin{aligned}\widehat{|3\rangle} &= |3\rangle + \frac{\langle 45 \rangle}{\langle 35 \rangle} |4\rangle, \\ \widehat{P} &= p_4 + p_5 = |5\rangle \left(\frac{\langle 34 \rangle}{\langle 35 \rangle} [4] + [5] \right).\end{aligned}\tag{5.5}$$

The amplitude $A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^+)$ is calculated from the shift “ $\langle 43 \rangle$ ”,

$$\begin{aligned}
A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^+) &= \left(\langle 4|1|5 \rangle + \frac{2p_2 \cdot p_3}{\langle 3|2|4 \rangle} \langle 3|1|5 \rangle \right) \\
&\times \left[\frac{[15](m\langle 43 \rangle[42] + 2p_1 \cdot p_5 \langle 32 \rangle) + ([14] + \frac{2p_2 \cdot p_3}{\langle 3|2|4 \rangle} [13]) (\langle 3|2|5 \rangle [42] + m[54] \langle 32 \rangle)}{8p_1 \cdot p_5 p_2 \cdot p_3 (p_4 \cdot p_5 + \frac{p_2 \cdot p_3}{\langle 3|2|4 \rangle} \langle 3|5|4 \rangle) [34]} \right] \\
&- \frac{m\langle 34 \rangle^3}{\langle 45 \rangle (p_1 + p_2)^4 (2p_2 \cdot p_3 + \frac{\langle 45 \rangle}{\langle 35 \rangle} \langle 3|2|4 \rangle)} \times \left[[1\hat{3}][\hat{P}2] - [1\hat{P}][\hat{3}2] \right],
\end{aligned} \tag{5.6}$$

where the shifted spinors in the second term are the same as in eq. (5.5).

VI. $\bar{q}qgggg$ AMPLITUDES

The six-point amplitudes with zero- and one-positive-helicity gluon are presented first. We have

$$\begin{aligned}
A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^-, 6_g^-) &= -m \left(\langle 34 \rangle + \frac{2\hat{P} \cdot p_{\hat{5}}}{[4|\hat{P}|\hat{5}]} \langle 3\hat{5} \rangle \right) \times \frac{1}{2p_2 \cdot p_3 [43]^2 [45] 2\hat{P} \cdot p_{\hat{5}} [65] 2p_1 \cdot p_6} \\
&\times \left[([16] 2\hat{P} \cdot p_{\hat{5}} + m[15] \langle 65 \rangle) m[43][42] \right. \\
&\quad \left. + (-[15][4|1|6] + m[16][54]) ([3|\hat{P}|\hat{5}] [42] - [4|\hat{P}|\hat{5}] [32]) \right],
\end{aligned} \tag{6.1}$$

where the shift “ $\langle 56 \rangle$ ” is used, so that

$$\begin{aligned}
|\hat{5}\rangle &= |5\rangle + \frac{2p_1 \cdot p_6}{[5|1|6]} |6\rangle, \quad |\hat{6}\rangle = |6\rangle - \frac{2p_1 \cdot p_6}{[5|1|6]} |5\rangle, \\
[a|\hat{P}|\hat{5}\rangle &= [a|2 - 3 - 4|\hat{5}\rangle, \\
\hat{P} \cdot p_{\hat{5}} &= -p_2 \cdot p_3 - p_2 \cdot p_4 + p_3 \cdot p_4.
\end{aligned} \tag{6.2}$$

We also use the “ $\langle 56 \rangle$ ” shift to obtain $A_6(\bar{1}_q^-, 2_q^-, 3_g^+, 4_g^-, 5_g^-, 6_g^-)$. This gives multiple non-vanishing diagrams as shown in fig. 5 which we denote as

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^+, 4_g^-, 5_g^-, 6_g^-) = \mathcal{D}_1^{+---} + \mathcal{D}_2^{+---} + \mathcal{D}_3^{+---}, \tag{6.3}$$

where $\mathcal{D}_1^{+---} = 0$, and the second diagram under the “ $\langle 56 \rangle$ ” shift yields

$$\mathcal{D}_2^{+---} = -m \frac{\langle 4\hat{5} \rangle^3}{\langle \hat{5}|\hat{P}|\hat{6}\rangle \langle 3|\hat{P}|\hat{6}\rangle \langle 34 \rangle p_{3,4,5}^2 2p_1 \cdot p_{\hat{6}}} \times \left[[1|\hat{P}_R|6\rangle [\hat{6}2] + [1\hat{6}\rangle \langle 6|\hat{P}_R|2] \right], \tag{6.4}$$

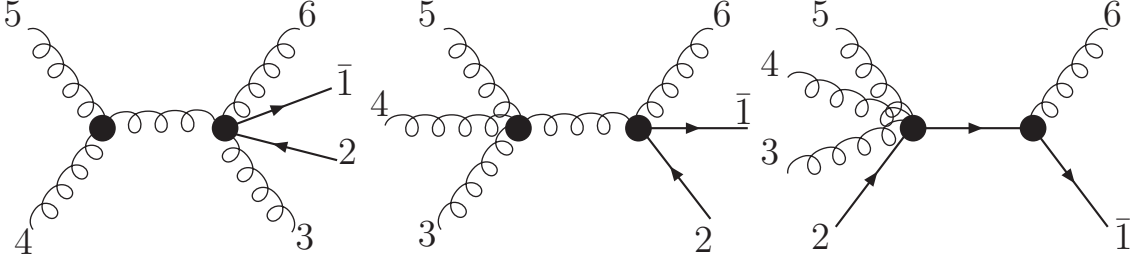


FIG. 5: The BCFW recursion diagrams required for the six-point amplitudes under the shifts “ $\langle 56 \rangle$ ” or “ $\langle 65 \rangle$ ”. This set of diagrams is relevant for the amplitudes with gluon helicities $(---)$, $(+---)$, $(-+--)$, $(-++-)$, and $(-+-+)$.

where $z = -\frac{p_{3,4,5}^2}{\langle 6|3+4|5 \rangle}$ so that

$$\begin{aligned}
 |\widehat{5}\rangle &= |5\rangle - \frac{p_{3,4,5}^2}{\langle 6|3+4|5 \rangle} |6\rangle, & |\widehat{6}\rangle &= |6\rangle + \frac{p_{3,4,5}^2}{\langle 6|3+4|5 \rangle} |5\rangle, \\
 \langle a|\widehat{P}|\widehat{6}\rangle &= -\langle a|(1+2)|\widehat{6}\rangle, \\
 \langle 6|\widehat{P}_R|b\rangle &= \langle 63\rangle[3b] + \langle 64\rangle[4b] + \langle 65\rangle[5b].
 \end{aligned} \tag{6.5}$$

For the third diagram, \mathcal{D}_3^{+---} , we find

$$\begin{aligned}
 \mathcal{D}_3^{+---} &= \left([3|2|4\rangle + \frac{2\widehat{P} \cdot p_{\widehat{5}}}{[4|\widehat{P}|\widehat{5}]} [3|2|\widehat{5}\rangle \right) \\
 &\times \frac{1}{2p_2 \cdot p_3 \left(2p_3 \cdot p_4 + \frac{2\widehat{P} \cdot p_{\widehat{5}}}{[4|\widehat{P}|\widehat{5}]} [4|3|\widehat{5}\rangle \right) 2\widehat{P} \cdot p_{\widehat{5}} [54] 2p_1 \cdot p_6 [65]} \\
 &\times \left[\left([3|\widehat{P}|\widehat{5}\rangle (-[15][4|1|6] + m[16][54]) + m[43]([16]2\widehat{P} \cdot p_{\widehat{5}} + m[15]\langle 65 \rangle) \right) \right. \\
 &\times \left(\langle 42 \rangle + \frac{2\widehat{P} \cdot p_{\widehat{5}}}{[4|\widehat{P}|\widehat{5}]} \langle \widehat{5}2 \rangle \right) \\
 &\left. - ([16]2\widehat{P} \cdot p_{\widehat{5}} + m[15]\langle 65 \rangle) 2p_2 \cdot p_3 [32] + m(-[4|1|6][15] + m[54][16]) \langle \widehat{5}4 \rangle [32] \right] \\
 &+ m \frac{\langle 4\widehat{5} \rangle^3}{[65] 2p_1 \cdot p_6 \langle 43 \rangle (p_1 + p_2 - p_{\widehat{6}})^4 (2p_1 \cdot p_{\widehat{5}} - 2p_5 \cdot p_6 + \frac{\langle 34 \rangle}{\langle 35 \rangle} (\langle \widehat{5}1|4\rangle - \langle 5|\widehat{6}|4 \rangle))} \\
 &\times \left[(m[16](\langle 53 \rangle + \frac{\langle 4\widehat{5} \rangle}{\langle 35 \rangle} [54]) - [15](\langle 6|1|3\rangle + \frac{\langle 4\widehat{5} \rangle}{\langle 35 \rangle} \langle 6|1|4 \rangle)) ([52] + \frac{\langle 34 \rangle}{\langle 35 \rangle} [42]) \right. \\
 &\left. - (m[16] \frac{\langle 34 \rangle}{\langle 35 \rangle} [54] - [15](\langle 6|1|5\rangle + \frac{\langle 34 \rangle}{\langle 35 \rangle} \langle 6|1|4 \rangle)) ([32] + \frac{\langle 4\widehat{5} \rangle}{\langle 35 \rangle} [42]) \right].
 \end{aligned} \tag{6.6}$$

In \mathcal{D}_3^{+---} we have,

$$\begin{aligned}\widehat{5} &= |5\rangle + \frac{2p_1 \cdot p_6}{[5|1|6]} |6\rangle, & \widehat{6} &= |6\rangle - \frac{2p_1 \cdot p_6}{[5|1|6]} |5\rangle, \\ \widehat{P} \cdot p_{\widehat{5}} &= -p_2 \cdot p_3 - p_2 \cdot p_4 + p_3 \cdot p_4, \\ [a|\widehat{P}|\widehat{5}] &= [a|2-3-4|\widehat{5}].\end{aligned}\tag{6.7}$$

From the shift “ $\langle 56 \rangle$ ” we have

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+, 5_g^-, 6_g^-) = \mathcal{D}_1^{-+--} + \mathcal{D}_2^{-+--} + \mathcal{D}_3^{-+--},\tag{6.8}$$

where the first diagram in fig. 5 is

$$\begin{aligned}\mathcal{D}_1^{-+--} &= -m \left(\frac{\langle 56 \rangle}{\langle 46 \rangle} \right)^3 \times \frac{\langle 34 \rangle + \frac{2p_1 \cdot p_6}{[\widehat{P}_R|1|6]} \langle 36 \rangle}{\langle 54 \rangle 2p_2 \cdot p_3 \left(\frac{p_{3,4,5}^2}{\langle 43 \rangle} \right)^2 \frac{p_{4,5,6}^2}{\langle 64 \rangle} 2p_1 \cdot p_{\widehat{6}}} \\ &\times \left[m[16][\widehat{P}_R 3][\widehat{P}_R 2] - [1\widehat{P}_R][3|1|6][\widehat{P}_R 2] + [1\widehat{P}_R][\widehat{P}_R|1|6][32] \right],\end{aligned}\tag{6.9}$$

with $|\widehat{P}_R\rangle = |4\rangle + \frac{\langle 56 \rangle}{\langle 46 \rangle} |5\rangle$. The second diagram yields

$$\mathcal{D}_2^{-+--} = -m \frac{\langle 3\widehat{5} \rangle^4}{\langle 34 \rangle \langle 4\widehat{5} \rangle \langle \widehat{5}|\widehat{P}|\widehat{6} \rangle \langle 3|\widehat{P}|\widehat{6} \rangle p_{3,4,5}^2 2p_1 \cdot p_{\widehat{6}}} \times \left[[1|\widehat{P}_R|6][\widehat{6}2] + [1\widehat{6}]\langle 6|\widehat{P}_R|2] \right]\tag{6.10}$$

and the third diagram

$$\begin{aligned}\mathcal{D}_3^{-+--} &= \frac{[4|\widehat{P}|\widehat{5}]}{2\widehat{P} \cdot p_{\widehat{5}} 2p_2 \cdot p_3 \left(2p_4 \cdot p_{\widehat{5}} + \frac{2p_2 p_3}{[4|2|3]} [4|\widehat{5}|3] \right) [34] 2p_1 \cdot p_6 [65]} \\ &\times \left[(-[15][4|1|6] + m[16][54]) ([4|\widehat{P}|\widehat{5}]\langle 32 \rangle + m\langle \widehat{5}3 \rangle [42]) \right. \\ &\quad \left. + ([16] 2\widehat{P} \cdot p_{\widehat{5}} + m[15]\langle 65 \rangle) [4|2|3][42] \right] \\ &+ \frac{m\langle 3\widehat{5} \rangle^3}{\langle \widehat{5}4 \rangle \langle 34 \rangle (p_1 + p_2 - p_6)^4 (2p_2 \cdot p_3 + \frac{\langle 4\widehat{5} \rangle}{\langle 3\widehat{5} \rangle} \langle 3|2|4 \rangle) 2p_1 \cdot p_6 [65]} \\ &\times \left[\left(-[15](\langle 6|1|3 \rangle + \frac{\langle 4\widehat{5} \rangle}{\langle 3\widehat{5} \rangle} \langle 6|1|4 \rangle) + m[16](\langle 53 \rangle + \frac{\langle 4\widehat{5} \rangle}{\langle 3\widehat{5} \rangle} [54]) \right) \left([52] + \frac{\langle 34 \rangle}{\langle 3\widehat{5} \rangle} [42] \right) \right. \\ &\quad \left. - \left(-[15](\langle 6|1|5 \rangle + \frac{\langle 34 \rangle}{\langle 3\widehat{5} \rangle} \langle 6|1|4 \rangle) + m[16] \frac{\langle 34 \rangle}{\langle 3\widehat{5} \rangle} [54] \right) \left([32] + \frac{\langle 4\widehat{5} \rangle}{\langle 3\widehat{5} \rangle} [42] \right) \right],\end{aligned}\tag{6.11}$$

where the spinor shifts in \mathcal{D}_2^{-+--} and \mathcal{D}_3^{-+--} are the same as eqs. (6.5) and (6.7), respectively.

Choosing the shift “ $\langle 43 \rangle$ ” leads to the diagrams in fig. 6,

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^+, 6_g^-) = \mathcal{D}_1^{---+} + \mathcal{D}_2^{---+} + \mathcal{D}_3^{---+}, \quad (6.12)$$

where we find

$$\begin{aligned} \mathcal{D}_1^{---+} = & \frac{[5|1|6\rangle}{2p_1 \cdot p_6 2\hat{P} \cdot p_4 \left(2p_5 \cdot p_6 + \frac{2\hat{P} \cdot p_4}{[5|\hat{P}|4]} [5|6|\hat{4}] \right) [45] 2p_2 \cdot p_3 [34]} \\ & \times \left[([16][5|\hat{P}|\hat{4}] + m[15]\langle 6\hat{4} \rangle) ([5|2|3][42] + m[54]\langle 32 \rangle) \right. \\ & \quad \left. - [15][5|1|6](2\hat{P} \cdot p_4 \langle 32 \rangle + m\langle 43 \rangle [42]) \right] \\ & + m \frac{\langle \hat{46} \rangle^4}{\langle 65 \rangle \langle \hat{45} \rangle (p_1 + p_2 - p_3)^4 (2\hat{P} \cdot p_4 + \frac{\langle 56 \rangle}{\langle \hat{46} \rangle} \langle \hat{4}|\hat{P}|5 \rangle) [34] 2p_2 \cdot p_3} \\ & \times \left[\left([14] + \frac{\langle 56 \rangle}{\langle \hat{46} \rangle} [15] \right) \left((\langle 3|2|6 \rangle + \frac{\langle \hat{45} \rangle}{\langle \hat{46} \rangle} \langle 3|2|5 \rangle) [42] + m([64] + \frac{\langle \hat{45} \rangle}{\langle \hat{46} \rangle} [54]) \langle 32 \rangle \right) \right. \\ & \quad \left. - \left([16] + \frac{\langle \hat{45} \rangle}{\langle \hat{46} \rangle} [15] \right) \left((\langle 3|2|4 \rangle + \frac{\langle 56 \rangle}{\langle \hat{46} \rangle} \langle 3|2|5 \rangle) [42] + m \frac{\langle 56 \rangle}{\langle \hat{46} \rangle} \langle 32 \rangle \right) \right]. \end{aligned} \quad (6.13)$$

The shifts for \mathcal{D}_1^{---+} are

$$\begin{aligned} |\hat{4}\rangle &= |4\rangle + \frac{2p_2 \cdot p_3}{[4|2|3]} |3\rangle, \quad |\hat{3}\rangle = |3\rangle - \frac{2p_2 \cdot p_3}{[4|2|3]} |4\rangle, \\ [5|\hat{P}|\hat{4}\rangle &= [5|6-1|\hat{4}\rangle, \\ \hat{P} \cdot p_4 &= p_1 \cdot p_5 + p_1 \cdot p_6 - p_5 \cdot p_6. \end{aligned} \quad (6.14)$$

The second diagram, \mathcal{D}_2^{---+} , is

$$\mathcal{D}_2^{---+} = -m \frac{\langle \hat{46} \rangle^4}{\langle \hat{45} \rangle \langle 56 \rangle \langle 6|\hat{P}|\hat{3}\rangle \langle \hat{4}|\hat{P}|\hat{3}\rangle 2p_2 \cdot p_3 p_{4,5,6}^2} \times \left[[1\hat{3}]\langle 3|\hat{P}|2\rangle + [1|\hat{P}|3\rangle \langle \hat{3}2] \right], \quad (6.15)$$

where \mathcal{D}_2^{---+} has $z = -\frac{p_{4,5,6}^2}{\langle 3|5+6|4\rangle}$ and

$$\langle 3|\hat{P}|b\rangle = \langle 34\rangle [4b] + \langle 35\rangle [5b] + \langle 36\rangle [6b]. \quad (6.16)$$

For the third diagram we have

$$\begin{aligned} \mathcal{D}_3^{---+} = & -m \frac{\langle 35 \rangle + \frac{2p_1 \cdot p_6}{\langle 6|1|\hat{P}]} \langle 36 \rangle}{2p_2 \cdot p_3 p_{3,4,5}^4 [\hat{P}6] 2p_1 \cdot p_6 \langle 54 \rangle} \times \frac{\langle 34 \rangle^3}{\langle 35 \rangle} \\ & \times \left[m[16] \frac{p_{3,4,5}^2}{\langle 35 \rangle} [\hat{P}2] - [1\hat{P}]\langle 6|1|\hat{3}\rangle [\hat{P}2] + [1\hat{P}]\langle 6|1|\hat{P}\rangle [\hat{3}2] \right], \end{aligned} \quad (6.17)$$

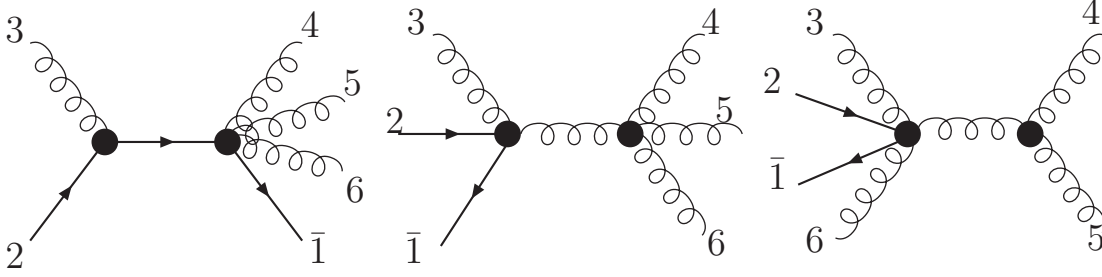


FIG. 6: The BCFW recursion diagrams used for the six-point amplitudes under the shifts “ $\langle 34 \rangle$ ” or “ $\langle 43 \rangle$ ”. These diagrams are applied for the amplitudes with gluon helicities $(--+-)$, $(---+)$, and $(- - + +)$.

where in the third diagram $z = -\frac{\langle 45 \rangle}{\langle 35 \rangle}$ and $|\widehat{P}] = |5] + \frac{\langle 34 \rangle}{\langle 35 \rangle}|4]$.

The “ $\langle 43 \rangle$ ” shift yields the final single-positive-helicity gluon amplitude at six points,

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^-, 6_g^+) = \mathcal{D}_1^{----+} + \mathcal{D}_2^{----+} + \mathcal{D}_3^{----+}, \quad (6.18)$$

where \mathcal{D}_3^{----+} vanishes. For the remaining diagrams, \mathcal{D}_1^{----+} and \mathcal{D}_2^{----+} , we have

$$\begin{aligned} \mathcal{D}_1^{----+} = & -\frac{[6|1|5] + \frac{2\widehat{P} \cdot p_4}{[5|\widehat{P}|4]}[6|1|4]}{2p_1 \cdot p_6 2\widehat{P} \cdot p_4 \left(2p_5 \cdot p_6 + \frac{2\widehat{P} \cdot p_4}{[5|\widehat{P}|4]}[5|6|4] \right) [45][34] (-2p_2 \cdot p_3)} \\ & \times \left[\left([16]2p_1 \cdot p_6 + m \left([15] + \frac{2\widehat{P} \cdot p_4}{[5|\widehat{P}|4]}[14] \right) [65] \right) \left(2\widehat{P} \cdot p_4 \langle 32 \rangle + m \langle 43 \rangle [42] \right) \right. \\ & \left. + \left(\left([15] + \frac{2\widehat{P} \cdot p_4}{[5|\widehat{P}|4]}[14] \right) [6|\widehat{P}|4] + m[16]\langle 54 \rangle \right) \left([5|2|3][42] + m[54]\langle 32 \rangle \right) \right] \\ & + m \frac{\langle 45 \rangle^3}{\langle 65 \rangle (p_1 + p_2 - p_3)^4 (2\widehat{P} \cdot p_4 + \frac{\langle 56 \rangle}{\langle 46 \rangle} \langle 4|\widehat{P}|5]) [34] 2p_2 \cdot p_3} \times \\ & \left[\left([14] + \frac{\langle 56 \rangle}{\langle 46 \rangle} [15] \right) \left((\langle 3|2|6] + \frac{\langle 45 \rangle}{\langle 46 \rangle} \langle 3|2|5]) [42] + m([64] + \frac{\langle 45 \rangle}{\langle 46 \rangle} [54]) \langle 32 \rangle \right) \right. \\ & \left. - \left([16] + \frac{\langle 45 \rangle}{\langle 46 \rangle} [15] \right) \left((\langle 3|2|4] + \frac{\langle 56 \rangle}{\langle 46 \rangle} \langle 3|2|5]) [42] + m \frac{\langle 56 \rangle}{\langle 46 \rangle} \langle 32 \rangle \right) \right] \end{aligned} \quad (6.19)$$

and the second diagram

$$\mathcal{D}_2^{----+} = -m \frac{\langle 45 \rangle^3}{\langle 56 \rangle \langle 6|\widehat{P}|3] \langle 4|\widehat{P}|3] 2p_2 \cdot p_3 p_{4,5,6}^2} \times \left[[1\widehat{3}] \langle 3|\widehat{P}|2] + [1|\widehat{P}|3] \langle 32] \right]. \quad (6.20)$$

The shifted spinors in the first and second diagrams, (6.19) and (6.20), are identical to those found in $A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^+, 6_g^-)$, eqs. (6.14) and (6.16) respectively.

Finally we write the amplitudes with two negative-helicity gluons, beginning with the amplitude

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^-, 5_g^+, 6_g^+) = \mathcal{D}_1^{---+} + \mathcal{D}_2^{---+} + \mathcal{D}_3^{---+}, \quad (6.21)$$

where the “ $\langle 34 \rangle$ ” shift is chosen. The first diagram, \mathcal{D}_1^{---+} , is

$$\begin{aligned} \mathcal{D}_1^{---+} = & \frac{\left(\langle 4|\hat{P}|5 \rangle + \frac{2p_1 \cdot p_6}{\langle 5|1|6 \rangle} \langle 4|\hat{P}|6 \rangle \right)}{2\hat{P} \cdot p_{\hat{4}} \left(2p_{\hat{4}} \cdot p_5 + \frac{2p_1 \cdot p_6}{\langle 5|1|6 \rangle} \langle 5|\hat{4}|6 \rangle \right) 2p_1 \cdot p_6 \langle 65 \rangle (-2p_2 \cdot p_3) [34]} \\ & \times \left[\left(-[15] \langle 4|1|6 \rangle + m[16] \langle 54 \rangle \right) \right. \\ & \times \left(\langle \hat{3}|2|5 \rangle [\hat{4}2] + m[5\hat{4}] \langle \hat{3}2 \rangle + \frac{2p_1 \cdot p_6}{\langle 5|1|6 \rangle} \left(\langle \hat{3}|2|6 \rangle [\hat{4}2] + m[6\hat{4}] \langle \hat{3}2 \rangle \right) \right) \\ & \left. + \left(-[16] 2\hat{P} \cdot p_{\hat{4}} + m[15] [65] \right) \left(2\hat{P} \cdot p_{\hat{4}} \langle \hat{3}2 \rangle + m \langle 43 \rangle [\hat{4}2] \right) \right] \\ & - \frac{m[56]^3}{(p_1 + p_2 - p_3)^4 \left(2p_1 \cdot p_6 + \frac{\langle \hat{4}5 \rangle}{\langle \hat{4}6 \rangle} \langle 5|1|6 \rangle \right) [5\hat{4}] [43] 2p_2 \cdot p_3} \\ & \times \left[\left(([14] + \frac{[56]}{[\hat{4}6]} [15]) \left(\langle 6|\hat{P}|\hat{4} \rangle + \frac{[\hat{4}5]}{[\hat{4}6]} \langle 5|\hat{P}|\hat{4} \rangle \right) \right. \right. \\ & \left. \left. - \left([16] + \frac{[\hat{4}5]}{[\hat{4}6]} [15] \right) \left(2\hat{P} \cdot p_{\hat{4}} + \frac{[56]}{[\hat{4}6]} \langle 5|\hat{P}|\hat{4} \rangle \right) \right) \langle \hat{3}2 \rangle + m[1\hat{3}] \frac{p_{4,5,6}^2}{[\hat{4}6]} [\hat{4}2] \right]. \end{aligned} \quad (6.22)$$

For the diagram \mathcal{D}_1^{---+} the spinors are shifted with

$$\begin{aligned} z &= \frac{-2p_2 \cdot p_3}{\langle 4|2|3 \rangle}, \\ \langle 4|\hat{P}|5 \rangle &= \langle 4|6-1|5 \rangle, \quad \langle a|\hat{P}|\hat{4} \rangle = \langle a|5+6-1|\hat{4} \rangle, \\ \hat{P} \cdot p_{\hat{4}} &= p_1 \cdot p_5 + p_1 \cdot p_6 - p_5 \cdot p_6. \end{aligned} \quad (6.23)$$

We note that the second term in \mathcal{D}_1^{---+} arises from a vanishing term in $A_5(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+, 5_g^+)$. This five-point term is zero for the special quark spins chosen but contributes a residue at six points since the recursion sums over both spin states of an internal quark. The second diagram yields

$$\begin{aligned} \mathcal{D}_2^{---+} = & \frac{\langle \hat{3}|2|5 \rangle \langle 54 \rangle + \langle \hat{3}|2|6 \rangle \langle 64 \rangle}{(p_1 + p_2)^2 2p_2 \cdot p_{\hat{3}} p_{4,5,6}^2} \frac{[65]}{\langle 45 \rangle \langle 56 \rangle [5\hat{4}]} \\ & \times \left[[\hat{1}\hat{3}] (\langle 45 \rangle [52] + \langle 46 \rangle [62]) - ([15] \langle 54 \rangle + [16] \langle 64 \rangle) \langle \hat{3}2 \rangle \right], \end{aligned} \quad (6.24)$$

where the \widehat{P} spinors have been eliminated using $\langle a\widehat{P} \rangle [\widehat{P}b] = \langle a|\widehat{4} + 5 + 6|b\rangle$, and $z = \frac{p_{4,5,6}^2}{\langle 4|5+6|3\rangle}$. For the third diagram we have

$$\begin{aligned} \mathcal{D}_3^{-++} = & -\frac{\left(\langle \widehat{3}|2|5\rangle + \frac{2p_1 \cdot p_6}{\langle \widehat{P}|1|6\rangle} \langle \widehat{3}|2|6\rangle\right)}{2p_2 \cdot p_3 \left(2\widehat{P} \cdot p_3 + \frac{2p_1 \cdot p_6}{\langle \widehat{P}|1|6\rangle} \langle \widehat{P}|3|6\rangle\right)} \times \frac{[35]}{[45][34]} \\ & \times \left[\left(-[1\widehat{P}] \langle \widehat{3}|1|6\rangle + m[16] \langle \widehat{P}\widehat{3}\rangle\right) \left([52] + \frac{2p_1 \cdot p_6}{\langle \widehat{P}|1|6\rangle} [62]\right) - \left([16]2p_2 \cdot p_3 - m[1\widehat{P}] [65]\right) \langle \widehat{3}2 \rangle \right], \end{aligned} \quad (6.25)$$

where $z = \frac{[45]}{[35]}$. By using the Schouten identity we find

$$\begin{aligned} |\widehat{4}\rangle &= |4\rangle - \frac{[45]}{[35]} |3\rangle = \frac{[34]}{[35]} |5\rangle \\ \widehat{P} &= \left(\frac{[34]}{[35]} |4\rangle + |5\rangle\right) [5]. \end{aligned} \quad (6.26)$$

The next amplitude is

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+, 5_g^-, 6_g^+) = \mathcal{D}_1^{-++} + \mathcal{D}_2^{-++} + \mathcal{D}_3^{-++} \quad (6.27)$$

where the shift “ $\langle 65 \rangle$ ” is chosen to find

$$\begin{aligned} \mathcal{D}_1^{-++} = & \frac{\langle 3|2|4\rangle + \frac{2p_1 \cdot p_{\widehat{6}}}{\langle \widehat{P}_R|1|6\rangle} \langle 3|2|6\rangle}{2p_2 \cdot p_3 \left(2\widehat{P}_R \cdot p_3 + \frac{2p_1 \cdot p_{\widehat{6}}}{\langle \widehat{P}_R|1|6\rangle} \langle \widehat{P}_R|3|6\rangle\right)} \times \frac{[46]^2}{[54][56]} \\ & \times \left[\left(-[1\widehat{P}_R] \langle 3|1|6\rangle + m[16] \langle \widehat{P}_R 3\rangle\right) \left([42] + \frac{2p_1 \cdot p_{\widehat{6}}}{\langle \widehat{P}_R|1|6\rangle} [62]\right) \right. \\ & \left. + \left(-[16]2p_2 \cdot p_3 + m[1\widehat{P}_R] [64]\right) \langle 32 \rangle \right], \end{aligned} \quad (6.28)$$

where $z = \frac{[45]}{[46]}$, so then $\widehat{P}_R = \left(|4\rangle + \frac{[56]}{[46]} |5\rangle\right) [4]$ and $2\widehat{P}_R \cdot p_3 = p_{3,4,\widehat{5}}^2$. For the second diagram, \mathcal{D}_2^{-++} , we have

$$\begin{aligned} \mathcal{D}_2^{-++} = & -\frac{\langle 35 \rangle^4}{\langle 34 \rangle \langle 45 \rangle \langle 5|3+4|6 \rangle \langle 3|4+5|6 \rangle} \times \frac{1}{p_{3,4,5}^2 (p_1 + p_2)^2 2p_1 \cdot p_{\widehat{6}}} \\ & \times [6\widehat{P}] \langle \widehat{P}|2|6 \rangle \times \left[[1\widehat{P}] [\widehat{P}6] [62] - [16] [6\widehat{P}] \langle \widehat{P}2 \rangle \right] \end{aligned} \quad (6.29)$$

where $z = \frac{p_{3,4,5}^2}{\langle 5|3+4|6\rangle}$ and $|\widehat{P}\rangle[\widehat{P}6] = -(p_3 + p_4 + p_5)|6\rangle$. The third diagram is

$$\begin{aligned} \mathcal{D}_3^{-++-} &= \frac{\langle 5|6-1|4\rangle}{2\widehat{P} \cdot p_{\widehat{5}} 2p_2 \cdot p_3 \left(2p_4 \cdot p_{\widehat{5}} + \frac{2p_2 \cdot p_3}{\langle 3|2|4\rangle} \langle 3|\widehat{5}|4\rangle \right) [34] (-2p_1 \cdot p_6) \langle 56\rangle} \\ &\times \left[([16]\langle 5|\widehat{P}|4\rangle + m[15][64]) \left(m\langle 53\rangle[42] + \langle 5|\widehat{P}|4\rangle\langle 32\rangle \right) + [15]\langle 5|\widehat{P}|6\rangle\langle 3|2|4\rangle[42] \right] \\ &- m \frac{\langle 35\rangle^4}{\langle 56\rangle 2p_1 \cdot p_6 \langle 54\rangle \langle 34\rangle (p_1 + p_2 - p_{\widehat{6}})^4 (2p_2 \cdot p_3 + \frac{\langle 45\rangle}{\langle 35\rangle} \langle 3|2|4\rangle)} \\ &\times \left[(m[15](\langle 63\rangle + \frac{\langle 45\rangle}{\langle 35\rangle} [64]) + [16](\langle 5|6-1|3\rangle + \frac{\langle 45\rangle}{\langle 35\rangle} \langle 5|6-1|4\rangle)(\langle \widehat{5}2\rangle + \frac{\langle 34\rangle}{\langle 35\rangle} [42]) \right. \\ &\quad \left. - (m[15](\langle 65\rangle + \frac{\langle 34\rangle}{\langle 35\rangle} [64]) + [16](2\widehat{P} \cdot p_{\widehat{5}} + \frac{\langle 34\rangle}{\langle 35\rangle} \langle 5|6-1|4\rangle)([32] + \frac{\langle 45\rangle}{\langle 35\rangle} [42]) \right], \end{aligned} \quad (6.30)$$

where $z = -\frac{2p_1 \cdot p_6}{\langle 5|1|6\rangle}$. Then $\widehat{P} = p_{\widehat{6}} - p_1$ so that $\langle 5|\widehat{P}|b\rangle = \langle 5|6-1|b\rangle$ and $2\widehat{P} \cdot p_{\widehat{5}} = -2p_2 \cdot p_3 - 2p_2 \cdot p_4 + 2p_3 \cdot p_4$.

The final six-point amplitude with two negative-helicity gluons is found from the shift choice “ $\langle 56\rangle$ ”,

$$A_6(\bar{1}_q^-, 2_q^-, 3_g^-, 4_g^+, 5_g^+, 6_g^-) = \mathcal{D}_1^{-++-} + \mathcal{D}_2^{-++-} + \mathcal{D}_3^{-++-}. \quad (6.31)$$

We find for the first diagram in fig. 5

$$\begin{aligned} \mathcal{D}_1^{-++-} &= \frac{\langle 46\rangle}{\langle 56\rangle \langle 45\rangle} \times \frac{\langle 6|1|\widehat{P}_R\rangle}{2p_1 \cdot p_{\widehat{6}} 2p_2 \cdot p_3 \left(2\widehat{P}_R \cdot p_{\widehat{6}} + \frac{2p_2 \cdot p_3}{\langle 3|2|\widehat{P}_R\rangle} \langle 3|\widehat{6}|\widehat{P}_R\rangle \right) [3\widehat{P}_R]} \\ &\times \left[m[1\widehat{P}_R]\langle 63\rangle[\widehat{P}_R 2] + [16]\langle 3|2|\widehat{P}_R\rangle[\widehat{P}_R 2] - [1\widehat{P}_R]\langle 6|1|\widehat{P}_R\rangle\langle 32\rangle \right] \\ &- m \frac{\langle 36\rangle^4}{\langle 34\rangle \langle 45\rangle \langle 56\rangle (p_1 + p_2)^4 (2p_2 \cdot p_3 + \frac{\langle 46\rangle}{\langle 36\rangle} \langle 3|2|\widehat{P}_R\rangle)} \\ &\times \left[([13] + \frac{\langle 46\rangle}{\langle 36\rangle} [1\widehat{P}_R])(\langle \widehat{6}2\rangle + \frac{\langle 34\rangle}{\langle 36\rangle} [\widehat{P}_R 2]) - ([1\widehat{6}] + \frac{\langle 34\rangle}{\langle 36\rangle} [1\widehat{P}_R])([32] + \frac{\langle 46\rangle}{\langle 36\rangle} [\widehat{P}_R 2]) \right], \end{aligned} \quad (6.32)$$

where $z = -\frac{\langle 45\rangle}{\langle 46\rangle}$ and $\widehat{P}_R = |4\rangle \left([4] + \frac{\langle 56\rangle}{\langle 46\rangle} [5] \right)$. The second diagram is

$$\begin{aligned} \mathcal{D}_2^{-++-} &= -\frac{[45]^3}{\langle 6|\widehat{P}|5\rangle \langle 6|\widehat{P}|3\rangle [34]} \times \frac{1}{p_{3,4,5}^2 2\widehat{P} \cdot p_{\widehat{6}} 2p_1 \cdot p_{\widehat{6}}} \\ &\times \langle 6|2|\widehat{P}\rangle \langle \widehat{P}6\rangle \times \left[-[1\widehat{P}]\langle \widehat{P}6\rangle \langle 62\rangle + [16]\langle 6\widehat{P}\rangle[\widehat{P}2] \right], \end{aligned} \quad (6.33)$$

where $z = -\frac{p_{3,4,5}^2}{\langle 6|3+4|5\rangle}$, $-2\hat{P} \cdot p_{\hat{6}} = (p_1 + p_2)^2$, and $|\hat{P}\rangle\langle\hat{P}6\rangle = -(p_3 + p_4 + p_5)|6\rangle$. The third diagram, \mathcal{D}_3^{-++-} , is

$$\begin{aligned} \mathcal{D}_3^{-++-} = & \frac{\langle 3|2|4\rangle + \frac{2\hat{P} \cdot p_{\hat{5}}}{\langle 4|\hat{P}|5\rangle} \langle 3|2|5\rangle}{2p_2 \cdot p_3 \left(2p_3 \cdot p_4 + \frac{2\hat{P} \cdot p_{\hat{5}}}{\langle 4|\hat{P}|5\rangle} \langle 4|3|5\rangle \right) 2\hat{P} \cdot p_{\hat{5}} \langle \hat{5}4\rangle [65] 2p_1 \cdot p_6} \\ & \times \left[[16]\langle 4|\hat{P}|5\rangle \left(\langle 3|\hat{P}|5\rangle \left([42] + \frac{2\hat{P} \cdot p_{\hat{5}}}{\langle 4|\hat{P}|5\rangle} [52] \right) + m[54]\langle 32\rangle \right) \right. \\ & + [15] \left(\langle 6|\hat{P}|5\rangle \left(m\langle 43\rangle \left([42] + \frac{2\hat{P} \cdot p_{\hat{5}}}{\langle 4|\hat{P}|5\rangle} [52] \right) - 2p_2 \cdot p_3 \langle 32\rangle \right) \right. \\ & \left. \left. + m\langle 64\rangle \left(\langle 3|\hat{P}|5\rangle \left([42] + \frac{2\hat{P} \cdot p_{\hat{5}}}{\langle 4|\hat{P}|5\rangle} [52] \right) + m[54]\langle 32\rangle \right) \right) \right] \\ & - \frac{m[45]^3}{(p_1 + p_2 - p_{\hat{6}})^4 \left(-2\hat{P} \cdot p_{\hat{5}} - \frac{[34]}{[35]} \langle 4|\hat{P}|5\rangle \right) [43][56] 2p_1 \cdot p_6} \\ & \times \left[[16] \left(\left(\langle 3|\hat{P}|5\rangle + \frac{[45]}{[35]} \langle 4|\hat{P}|5\rangle \right) \left(\langle \hat{5}2\rangle + \frac{[34]}{[35]} \langle 42\rangle \right) \right. \right. \\ & \left. \left. - \left(2\hat{P} \cdot p_{\hat{5}} + \frac{[34]}{[35]} \langle 4|\hat{P}|5\rangle \right) \left(\langle 32\rangle + \frac{[45]}{[35]} \langle 42\rangle \right) \right) + m[15] \frac{p_{3,4,\hat{5}}^2}{[35]} \langle 62\rangle \right], \end{aligned} \tag{6.34}$$

where $z = \frac{2p_1 \cdot p_6}{\langle 6|1|5\rangle}$, $2\hat{P} \cdot p_{\hat{5}} = (p_2 - p_3 - p_4)^2$, and $\langle a|\hat{P}|5\rangle = \langle a|6-1|5\rangle$.

VII. CONCLUSIONS

We used the BCFW formalism to calculate the scattering amplitudes with a pair of massive fermions and four or fewer gluons by shifting adjacent pairs of massless gluons. All the helicity configurations are computed in this way. By deriving an explicit formula for the shifted spinors of an internal gluon, no subtleties are encountered [21] with the massive-spinor inner products.

We have performed a number of checks on our results. By performing a variety of shifts we numerically confirmed that the residues corresponding to all possible channels are correct. We note that shifts where the amplitude is not well behaved for large z can be used for the purpose of checking a putative amplitude, even though an on-shell recursion constructed from such a shift would drop terms coming from the additional large z contribution. To perform the check we shift a different pair of spinors than the ones used to construct the amplitude via on-shell recursion and extract each residue which we then compare to the

corresponding BCFW diagram. This verifies that the answers are consistent, so that we obtain the proper residues no matter which pair of external gluons are shifted. This check is equivalent to checking all factorization limits of the amplitude.

We have also checked that our results agree with the amplitudes presented in ref. [40], where formulas for the amplitudes with identical or all but one identical helicity gluons were derived from massive scalar amplitudes via supersymmetric Ward identities. The five-point amplitudes match the results of ref. [21], which were calculated using BCFW recursion and Feynman diagrams for certain amplitudes.

The amplitudes presented in this paper are a subset of those needed for heavy quark plus two-jet production at the LHC. The calculations of this paper confirm that the BCFW recursion relations are an effective way to compute amplitudes with massive quarks.

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